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ON THE GEOMETRY OF A TOTALLY DAMAGED ZONE NEAR A MODE III CRACK TIP IN CREEP-DAMAGE COUPLED PROBLEM

The asymptotic solution to a mode III crack growth problem in a creeping solid in the framework of continuum damage mechanics is presented. The kinetic law of damage evolution is the Kachanov – Rabotnov equation. The damage parameter is introduced in the power stress – strain-rate constitutive relations. Thus, the coupled system of continuum damage mechanics – creep theory equations is considered. The solution is based on introducing a self – similar variable proposed by Riedel for the creep power constitutive law coupled with the power kinetic law of damage evolution. The present contribution is an attempt to obtain the stress, strain rate and damage distributions in the vicinity of the crack tip as the dimensionless functions of the similarity variable under the assumption of the existence of a totally damaged zone near the crack tip. It is supposed that there is the totally damaged zone in the vicinity of the crack tip where the damage parameter reaches its critical value and the stresses are equalled to zero. The shape and the characteristic length of the totally damaged zone are not known and should be found as a part of the solution. Since the governing system of equations is not valid inside of the totally damaged zone, an analytical solution for the stress and damage fields is sought at large distances from the crack tip. The three – term asymptotic expansion of the effective stress (the stress referred to the surface that really transmits the internal forces) and the four – term asymptotic expansion of the integrity (continuity) parameter for large distances from the crack tip are obtained. It is found that the HRR-solution for the creep power constitutive law can not be used as the remote boundary condition and the actual stress field at infinity is determined. The construction technique of the far-field stress asymptotics is elucidated. The asymptotic fields allow to obtain the configuration of the totally damaged zone modelled in the vicinity of the crack. It is shown that the totally damaged zone near of the crack tip does really exist. The geometry of the totally damaged zone for different values of the material constants is numerically studied and given.

1. Introduction.

For some forty years, a great deal of research has been going on in the concepts of continuum damage mechanics, originally introduced by Rabotnov [1] and Kachanov [2]. Recently a number of papers devoted to the static and growing crack problems coupling elasticity, plasticity and creep with damage in the framework of continuum damage mechanics have increased [3-8]. Effects of the damage evolution on the stress – strain state and vice versa are of interest to study. Some characteristic features related to the two – dimensional damage coupled crack problems can be outlined.

It is shown in [3-8] that the effect of damage evolution on the stress – strain state in the vicinity of the crack tip appears as either smoothing the stress singularity near the crack tip or its significant reduction. The effective stress is limited at the crack tip and the integrity parameter is a linear function of the distance from the moving crack tip [3,4].The asymptotic analysis of the stress – strain state and the damage field near the fatigue growing crack with the general constitutive equations including the damage coupled stress – strain relation and the damage evolution equation shows [5] weaker stress singularity near the crack tip (in comparison with the classical square root stress singularity of linear fracture mechanics) whereas the stress singularity for some values of the material constants disappears entirely. It is found [6] that the stress field for a growing mode I crack in a material with the damage coupled power-law elastic-plastic constitutive equations has no singularity in the vicinity of the crack tip.

The following feature is the presence of a damage process zone or a totally damaged zone [3-8].The determination of the totally damaged zone or the active damage process zone's shapes is of considerable interest. The geometry of the progressively damaged zone in the vicinity of the fatigue growing crack is determined as the kinetic law has two branches [5]. The condition separating two regimes (the active damage accumulation and lack of this process) allows to obtain the contour of the zone. The asymptotic fields of stress, creep strain rate and damage of a mode I creep crack in steady-state growth are analyzed in [7,8]according to the experimental observation on the damage distribution around a mode I creep crack in steady-state growth. The contour of the damage field is represented by a semi-ellipse in front of the crack and by a wake parallel to the crack plane behind the crack. Thus, the problem of determining the configuration of the totally damaged zone or the progressively damaged zone still remains unsolved. The present paper is concerned with an asymptotic analysis of the stress, creep strain rate and the integrity parameter fields at large distances from the crack tip. The asymptotic analysis is based on a self – similar variable proposed by Riedel [9]. The asymptotic fields allow to obtain the configuration of the totally damaged zone modelled in the vicinity of the crack.

2. A self-similar variable of the creep crack growth in a solid with damage problem.

Let us consider a semi-infinite crack in an infinite body in a material with constitutive equations formulated on the basis of the creep power law in the framework of continuum damage mechanics

$$\dot{\varepsilon}_{ij} = \frac{3}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{s_{ij}}{\psi}, \tag{2.1}$$

where $\dot{\varepsilon}_{ij}$ is the creep strain rate; ψ is the integrity parameter; s_{ij} and $\sigma_e = \sqrt{3s_{ij}s_{ij}/2}$ are the deviatoric stress and the equivalent stress respectively. The symbols n and B are the creep exponent and material constant, respectively.

Asymptotic conditions at $r \to \infty$ have a more general form (as compared with the Hutchinson – Rice – Rosengren (HHR)-solution [10])

$$\sigma_{ij}(r,\theta,t) = \tilde{C}r^s\overline{\sigma}_{ij}(\theta), \qquad \psi(r,\theta,t) = 1 \qquad (r\to\infty), \tag{2.2}$$

where $\tilde{C}$ is the stress amplitude at infinity. The value of s and the angular distributions $\overline{\sigma}_{ij}(\theta)$ are to be found. It should be noted that analysis of the kinetic equation of damage evolution represented in a moving with the crack tip coordinate system in the dimensionless form

$$\cos\theta\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta} = \left(\frac{\sigma_e}{\psi}\right)^m \tag{2.3}$$

shows the following. If the stress and the integrity parameter asymptotic expansions are sought as

$$\sigma_{ij}(r,\theta) = r^s f_{ij}(\theta), \quad \psi(r,\theta) = 1 - r^\beta g(\theta), \quad (s,\ \beta<0) \quad (r\to\infty) \tag{2.4}$$

the exponents s and β are connected by the relation $\beta = 1 + sm$. Asymptotic condition of approaching to HHR – solution at infinity implies $s = -1/(n+1)$ and, consequently,

$\beta = (n+1-m)/m > 0$ $(m \approx 0.7)$. This is in contradiction with (2.4). Thus, the exponent s is to be found as a part of the solution.

It is found [9] that there is a self-similar variable

$$R = rk^{-1}t^{1/(sm)}, \qquad k = \left(A\tilde{C}\right)^{-1/(sm)}. \tag{2.5}$$

The symbols A, m are the constants of the damage evolution law

$$\frac{d\psi}{dt} = -A\left(\frac{\sigma_e}{\psi}\right)^m. \tag{2.6}$$

Equation (2.5) and the existence of the self-similar variable R are proved without any difficulties by dimensional analysis. For the case the stress and the integrity parameter are represented as

$$\sigma_{ij}(r,\theta,t) = \tilde{C}k^s t^{-1/m}\hat{\sigma}_{ij}(R,\theta), \qquad \psi(r,\theta,t) = \hat{\psi}(R,\theta), \tag{2.7}$$

where $\hat{\sigma}_{ij}(R,\theta)$ and $\hat{\psi}(R,\theta)$ are the dimensionless function of R,θ and should be obtained from the solution.

3. A growing mode III crack in a solid with damage. Creep-damage coupled problem.

The problem of mode III crack is considered under small-scale damage condition. The equilibrium equation has the form

$$\frac{\partial}{\partial R}\left(R\hat{\tau}_{Rz}\right) + \frac{\partial \hat{\tau}_{\theta z}}{\partial \theta} = 0, \tag{3.1}$$

where $\hat{\tau}_{iz}$ is the stress tensor components. The compatibility equation is formulated for the creep strain rate tensor components $\hat{\gamma}_{\theta z}$ and $\hat{\gamma}_{Rz}$,

$$\frac{\partial}{\partial R}\left(R\hat{\gamma}_{\theta z}\right) = \frac{\partial \hat{\gamma}_{Rz}}{\partial \theta}, \tag{3.2}$$

where

$$\hat{\gamma}_{Rz} = \left(\frac{\hat{\tau}}{\hat{\psi}}\right)^{n-1}\frac{\hat{\tau}_{Rz}}{\hat{\psi}}, \quad \hat{\gamma}_{\theta z} = \left(\frac{\hat{\tau}}{\hat{\psi}}\right)^{n-1}\frac{\hat{\tau}_{\theta z}}{\hat{\psi}}, \tag{3.3}$$

$$\hat{\gamma}_{ij}(R,\theta) = \frac{2\gamma_{ij}(r,\theta,t)}{3B}\left(\tilde{C}k^s t^{-1/m}\right)^{-n}, \qquad \tau = \sqrt{(\tau_{Rz})^2 + (\tau_{\theta z})^2}. \tag{3.4}$$

The kinetic law postulating the power law of damage evolution is written as

$$R\frac{\partial \hat{\psi}}{\partial R} = -sm\left(\frac{\hat{\tau}}{\hat{\psi}}\right)^m. \tag{3.5}$$

The solution of equations (3.1)–(3.5) should satisfy the traction-free surface condition at $\theta = \pi$ and the symmetry condition at $\theta = 0$

$$\hat{\tau}_{\theta z}(R,\theta=\pi) = 0, \quad \hat{\tau}_{Rz}(R,\theta=0) = 0. \tag{3.6}$$

Asymptotic condition at $R \to \infty$ has the form

$$\hat{\tau}_{ij}(R,\theta) = R^s \overline{\tau}_{ij}(\theta, n). \tag{3.7}$$

The solution to equations (3.1)–(3.5) subjected to the boundary conditions (3.6)–(3.7) is sought throughout except for the totally damaged zone near the crack tip where the stated system of equations is not valid. The shape of the totally damage zone is *a priori* unknown and shoud be found as a part of the solution. It is suggested that all stress and the integrity parameter in the totally damaged zone are equal to zero. The solution has to satisfy the continuity conditions $\hat{\psi} = 0$, $\hat{\tau}_{ij} = 0$ on a boundary of the totally damaged zone. Henceforth we shall omit the sign $\hat{}$.

4. The eigenfunction method.

One can find asymptotic expansions of the stresses and the integrity parameter at large distances from the crack tip R (at large distances in comparison with a characteristic length of the totally damaged zone, but at as yet small distances in comparison with a characteristic length of the body or the crack length). By virtue of the fact that the totally damaged zone near the crack tip does exist it is not possible to seek for the asymptotic expansions in the vicinity of the crack tip. This difficulty may be overcome by approaching to the totally damaged zone from infinity ($R \to \infty$).

Thus, the major terms of the stress tensor components and integrity parameter asymptotic expansions are sought in the form

$$\tau_{ij}(R,\theta) = R^s f_{ij}^{(0)}(\theta), \quad \psi(R,\theta) = 1 \quad (R \to \infty, s < 0). \tag{4.1}$$

Equations (3.1), (3.2) and (4.1) result in a system of the two ordinary differential equations with respect to $f_{Rz}^{(0)}(\theta)$ and $f_{\theta z}^{(0)}(\theta)$:

$$\frac{df_{\theta z}^{(0)}}{d\theta} + (s+1) f_{Rz}^{(0)} = 0, \qquad \frac{d}{d\theta}\left(f^{n-1} f_{Rz}^{(0)}\right) = (sn+1) f^{n-1} f_{\theta z}^{(0)}, \tag{4.2}$$

where $f = \sqrt{\left(f_{Rz}^{(0)}\right)^2 + \left(f_{\theta z}^{(0)}\right)^2}$. The solution to the system (4.2) should obey the traction-free surface conditions and the symmetry conditions:

$$f_{\theta z}^{(0)}(\theta = \pi) = 0, \qquad f_{Rz}^{(0)}(\theta = 0) = 0. \tag{4.3}$$

In view of homogeneity of the system (4.2) the functions $\kappa f_{ij}^{(0)}(\theta)$, where κ is an arbitrary multiplier, are the solution to the system if the functions $f_{ij}^{(0)}(\theta)$ are the solutions of equations (4.2) either. Consequently, one can accept the normalization condition

$$f_{\theta z}^{(0)}(0) = 1. \tag{4.4}$$

Analysis of the kinetic law (3.5) allows to find the second term of the integrity parameter asymptotic expansion. Substitution of the known first term of the equivalent effective stress asymptotic expansion

$$\frac{\tau}{\psi}(R,\theta) = R^s f(\theta) \tag{4.5}$$

into the kinetic law (3.5) leads to the equation

$$R\frac{\partial \psi}{\partial R} = -smR^{sm} f^m, \tag{4.6}$$

which can be integrated in respect with the self-similar variable R. Integrating the kinetic law and taking account the boundary condition at infinity one can derive the two-term asymptotic expansion for the integrity parameter:

$$\psi(R, \theta) = 1 - R^{sm} f^m(\theta). \tag{4.7}$$

The obtained two terms of the integrity parameter asymptotic expansion enable to determine the shape of the totally damage zone where the integrity parameter is equal to zero. The boundary of the zone is governed by the equation

$$\psi(R, \theta) = 1 - R^{sm} f^m(\theta) = 0 \qquad \text{or} \qquad R = R(\theta) = (f(\theta))^{-1/s}.$$

To refine the solution one can find the following terms of the effective stress and the integrity parameter asymptotic expansions. The effective stress asymptotic expansion is sought in form

$$\frac{\tau_{ij}}{\psi}(R, \theta) = R^s f_{ij}^{(0)}(\theta) + R^{s_1} f_{ij}^{(1)}(\theta) + \dots . \tag{4.8}$$

Analysis of all terms of the stress asymptotic expansion allows to find $s_1 = s + sm$.

In view of the two-term asymptotic expansion of the integrity parameter (4.7) the stress asymptotic expansion can be written as

$$\tau_{ij}(R, \theta) = R^s f_{ij}^{(0)}(\theta) + R^{s+sm} \left(f_{ij}^{(1)}(\theta) - f_{ij}^{(0)}(\theta) f^m(\theta) \right). \tag{4.9}$$

Substituting the stress asymptotic expansion (4.9) into the equilibrium equation (3.1) and the compatibility equation (3.2) and setting coefficients at the same degree of R equal to zero one can derive the system of the ordinary differential equations with to the first (major) term of the asymptotic expansion (4.2) and the second one (the system being cumbersome is not given here). Since

$$\left(\frac{\tau}{\psi}\right)^m = R^{ms} f^m \left(1 + R^{s_1 - s} m f_1 / f^2\right), \tag{4.10}$$

where $f_1 = f_{Rz}^{(0)} f_{Rz}^{(1)} + f_{\theta z}^{(0)} f_{\theta z}^{(1)}$, equation (3.5) can be rewritten as

$$R\frac{\partial \psi}{\partial R} = -sm \left(R^{sm} f^m + mR^{2sm} f^{m-2} f_1\right). \tag{4.11}$$

The latter expression enables to determine the following approximation for the boundary of the totally damaged zone:

$$\psi(R, \theta) = 1 - R^{sm} f^m - mR^{2sm} f^{m-2} f_1 / 2 = 0. \tag{4.12}$$

Therefore, the boundary of the totally damaged zone is determined by the relation

$$R = R(\theta) = \left[\left(f^m + \sqrt{f^{2m} + 2m f^{m-2} f_1} \right) / 2 \right]^{-1/(sm)}. \tag{4.13}$$

The three-term asymptotic expansion of the effective stress tensor is sought in the form

$$\frac{\tau_{ij}}{\psi}(R,\theta) = R^s f_{ij}^{(0)}(\theta) + R^{s_1} f_{ij}^{(1)}(\theta) + R^{s_2} f_{ij}^{(2)}(\theta). \tag{4.14}$$

Considering the integrity parameter asymptotic expansion obtained one can find that $s_2 = s + 2sm$, and the stress asymptotic expansion has the form

$$\begin{aligned}\tau_{ij}(R,\theta) = R^s f_{ij}^{(0)}(\theta) + R^{s+sm}\left(f_{ij}^{(1)}(\theta) - f_{ij}^{(0)}(\theta) f^m(\theta)\right) + \\ + R^{s+2sm}\left(f_{ij}^{(2)}(\theta) - f_{ij}^{(1)}(\theta) f^m(\theta) - m f_{ij}^{(0)}(\theta) f^{m-2}(\theta) f_1(\theta)\right).\end{aligned} \tag{4.15}$$

The kinetic law of damage evolution gives the fourth term of the integrity parameter asymptotic expansion and the equation determining the boundary of the totally damaged zone is found to be given by

$$\psi = 1 - R^{sm} f^m - mR^{2sm} f^{m-2} f_1/2 - mR^{3sm} f^{m-2} f_2/6 = 0. \tag{4.16}$$

The solution to equation (4.16) is numerically sought for different values of the material constants n, m.

$n = m = 1$	$s = -1.5$
$n = 2,\ m = 0.7n$	$s = -1.2303$
$n = 3,\ m = 0.7n$	$s = -1.1830$
$n = 4,\ m = 0.7n$	$s = -1.1648$
$n = 5,\ m = 0.7n$	$s = -1.1553$
$n = 6,\ m = 0.7n$	$s = -1.1495$
$n = 7,\ m = 0.7n$	$s = -1.1455$
$n = 8,\ m = 0.7n$	$s = -1.1425$
$n = 9,\ m = 0.7n$	$s = -1.1405$
$n = 10,\ m = 0.7n$	$s = -1.1390$

The eigenvalues computed for different values of the material constants n, $m = 0.7n$.

5. Conclusion.

1) It can be concluded the HRR-field does not govern the geometry of the totally damaged zone. The new stress asymptotic at infinity is found. It is shown that it is precisely the stress asymptotic which governs the geometry of the totally damaged zone for different values of the material constants.

2) The geometry of the totally damaged zone is found and given in Figure 1-4 where the following notations are accepted: 1 – the configuration of the totally damaged zone given by the two-term asymptotic expansion for the integrity parameter; 2 – the configuration of the totally damaged zone given by the three-term asymptotic expansion for the integrity parameter; 3 – the configuration of the totally damaged zone given by the four-term asymptotic expansion for the integrity parameter.

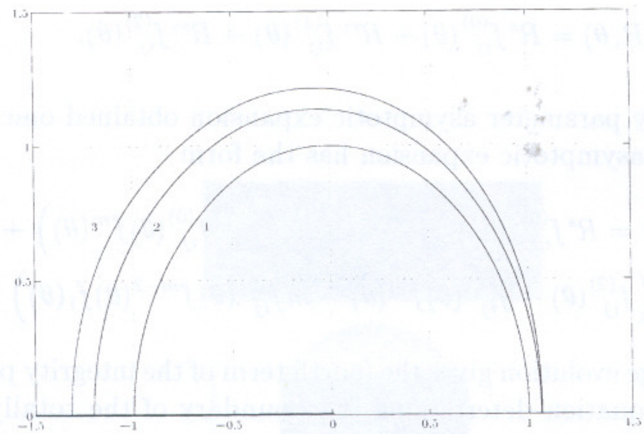

Figure 1. The geometry of the totally damaged zone for $n = m = 1\, m = 0.7n$.

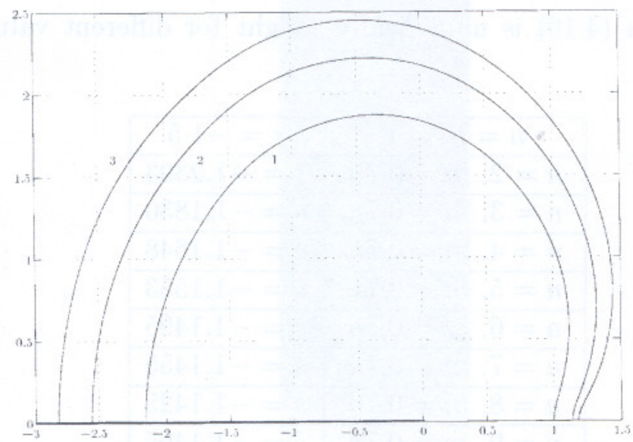

Figure 2. The geometry of the totally damaged zone for $n = 3$, $m = 0.7n$.

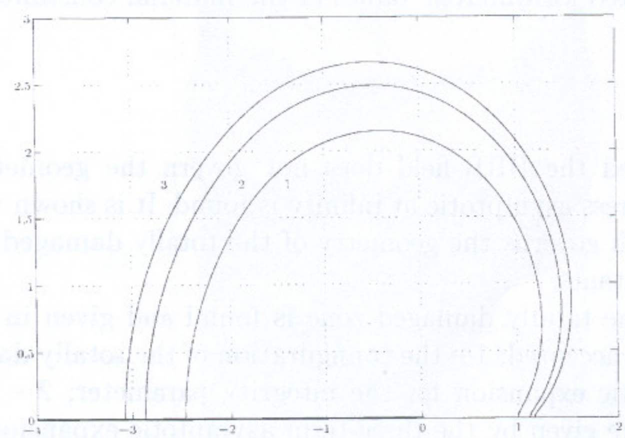

Figure 3. The geometry of the totaly damaged zone for $n = 5$ and $m = 0.7n$.

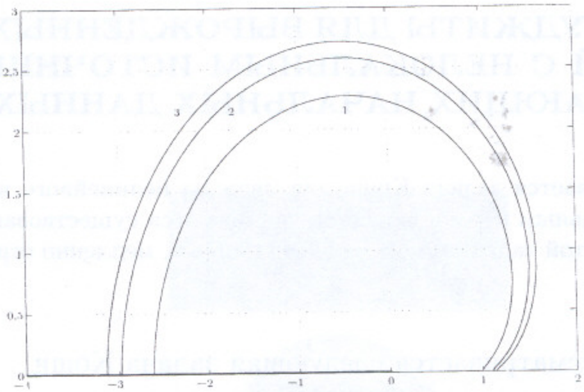

Figure 4. The geometry of the totaly damaged zone for $n = 7$ and $m = 0.7n$.

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